EXTENDED NOISE-SHAPING IN CASCADED N-TONE $\Sigma\Delta$ CONVERTERS

Shubha Bommalingaiahnapallya, Raghavendra Bommalingaiahnapallya and Ramesh Harjani
Department of Electrical and Computer Engineering
University of Minnesota, Minneapolis, MN 55455
Email: harjani@ece.umn.edu, Tel: +1 612-625-4032

Abstract—An N-tone $\Sigma\Delta$ converter is obtained by introducing N zeros in the noise transfer function (NTF) while maintaining a flat signal transfer function (STF). N-tone $\Sigma\Delta$ converters are particularly well suited for multi-carrier systems. While oversampling is a widely used technique to improve the resolution of the converter, different methods will have to be explored for systems with low over-sampling. Even though, the achievable resolution is limited by the number of bits per Nyquist sample, using higher-order loop-filters and multi-bit loop quantizers can be used to achieve high resolution. But, the former is prone to instability and the latter is sensitive to DAC linearity. However, use of cascaded converters has proven to overcome these inherent disadvantages of the before-mentioned methods. In this paper, we extend this existing low-pass technique to the N-Tone $\Sigma\Delta$ converters.

I. INTRODUCTION TO N-TONE $\Sigma\Delta$ CONVERTERS

Of late, many high-performance wireless and wireline systems employ multi-carrier orthogonal frequency division multiplexing (MC-OFDM) to obtain high performance with low system complexity. The N-Tone $\Sigma\Delta$ converter perfectly matches the N-tone MC-OFDM signal and hence, can utilize the narrow band-pass and multi-tone property of the signal [1] [2]. N-Tone $\Sigma\Delta$ Converters can also find potential application in cognitive radios, to sniff the frequency and then operate in an unused subset of the frequency domain. M-FSK systems can use a single optimized Analog-to-Digital converter (ADC) to look for RF energy in different frequency intervals.

MC-OFDM employs a train of modulated orthogonal narrow-band sub-carriers which not only offers high spectral efficiency but also multi-path resolution [3]. A look at the spectrum of the coded pulse-train (Figure 1) reveals that the signal is a multi-tone narrow band-pass signal which is best processed in the digital domain. And, for the most versatile communication system, the conversion to digital should be done as high up in the receiver chain as possible. It is also interesting to note that the spectrum of the multi-tone MC-OFDM signals are sinc-shaped. Hence, one could spread the zeros of the NTFs throughout the main lobe to make efficient use of the spectrum and obtain better SNR. Further, these MC-OFDM signals have low over-sampling ratios (OSR) to make efficient use of the spectrum.

Since, the system has low over-sampling, one has to use higher order loop-filters, multi-bit quantizers or cascaded architectures to obtain good SNR. This paper extends existing low-pass techniques used to improve the SNR of systems with low OSR to N-Tone $\Sigma\Delta$ Converters.

II. TECHNIQUES TO IMPROVE SNR IN N-TONE $\Sigma\Delta$ CONVERTERS

A $\Sigma\Delta$ converter consists of a loop filter and a quantizer in a feedback loop as shown in Figure 2. For a low-pass $\Sigma\Delta$ converter, the loop filter is an integrator. While the signal transfer function is all-pass, the noise transfer function is high-pass. Thus, quantization noise is effectively shaped away from the signal frequency. If the loop filter is replaced by an Nth-order resonator, there are N notches in the NTF. And, placing the multi-tone signal in these noise-valleys will result in a good SNR while maintaining low analog matching requirements.

Over-sampling is one technique that can be used to trade off resolution in time for resolution in amplitude with the use of simple analog components [4]. In a low-pass over-sampled $\Sigma\Delta$ converter, it is necessary for the sampling frequency to be much higher than the highest frequency component of the input to obtain a good SNR. But, for a band-pass $\Sigma\Delta$ converter, the sampling frequency need not be much higher than the bandwidth of the signal. Thus, for narrow-band signals, a relatively high performance can be achieved. But, increasing the over-sampling ratio in the MC-OFDM systems deteriorates the spectral usage due to which low over-sampling ratios are preferred.

One can employ higher order noise-shaping functions to get better SNR. For example, using an L-th order N-Tone
modulator, we see that we obtain an Lth other noise-shaping function of the form -

\[ Y(z) = X(z) \cdot z^{-2N} + E_q \cdot z^{-2N}(1 + z^{-2N})^L \]  

where \( X(z) \) represents the input, \( Y(z) \) represents the output and \( E_q \) represents the quantization noise. The noise power and maximum achievable SNR can be calculated to be given by

\[ P_n = \pi^{2L} N^{2L} \sigma_q^2 / (2L + 1)OSR^{2L+1} \]  

\[ SNR_{max} = 7.8 - 20 \cdot L \log(\pi N) + 10 \cdot \log((2L+1)OSR^{2L+1}) \]  

It seems that one can indefinitely improve the SNR by using higher order loop-filters. But, in fact, the resolution is actually limited by the number of output bits per Nyquist sample which is nothing but \( OSR \cdot N_{sub,ADC} \) where \( N_{sub,ADC} \) is the resolution of the quantizer within feedback. At low-oversampling ratios, the assumption that the quantization noise is white fails and the noise density in the desired band changes significantly so as to offset the advantage of using a higher order noise-shaping. Further, just like the higher order low-pass \( \Sigma\Delta \) converters, N-Tone \( \Sigma\Delta \) converters too suffer from instability. Further,

Use of multi-bit \( (N_{sub,ADC}) \) quantizers overcomes the output bits per Nyquist sample limit and allows for higher achievable SNR. Every extra bit added to the quantizer improves the SNR by about 6dB. However, the use of a multi-bit quantizer places stringent requirements on the DAC linearity. Any deviation from the ideal DAC levels can result in distortion. To avoid this, one could potentially employ tricks that are used in low-pass implementations such as multi-bit quantizers with a single-bit feedback [5].

To overcome the inherent problems of higher order loop-filters and multi-bit quantizers, cascaded modulators [6] which are basically a generalised version of the well-known MASH architectures of the \( \Sigma\Delta \) converters are used. The following section describes such architectures.

**III. CASCADEN ORDER N-TONE \( \Sigma\Delta \) CONVERTERS**

Shown in Figure 3 is the block diagram of the generalized 2-stage Cascaded N-tone \( \Sigma\Delta \) converter. As seen in the figure, the quantization noise of the first stage forms the input to the second stage. The second stage thus senses the quantization noise introduced by the first-stage quantizer. The "Digital Error Correction" block compensates the noise-shaping of \( Eq(z) \) in first stage and cancels it out. Thus, only the quantization noise of the second stage appears at the output. And, as this sees the noise shaping of both the stages, the SNR of the overall system is improved.

Let \( H_1(z) \) and \( N_1(z) \) be the STF and NTF of the first stage, then\[ Y_1(z) = X(z) \cdot H_1(z) + Eq(z) \cdot N_1(z) \]

Similarly, let \( H_2(z) \) and \( N_2(z) \) be the STF and NTF of the second stage, then\[ Y_2(z) = Eq(z) \cdot H_2(z) + Eq_2(z) \cdot N_2(z) \]

where \( Q_2(z) \) is the quantization noise of the second stage. After the digital error correction,

\[ Y(z) = Y_1(z) \cdot H_2^*(z) - Y_2(z) \cdot N_1^*(z) \]

Thus, the overall NTF, is the product of the NTFs of the cascaded stages. The effective NTF of ‘n’ stages in cascade is

\[ N(z) = N_1(z) \cdot N_2(z) \cdots N_n(z) \]

As predicted, the quantization noise of the first stage is cancelled completely only if the digital estimations - \( H_2^*(z) \) and \( N_1^*(z) \) - are equal to the actual values of the STF and NTF - \( H_2(z) \) and \( N_1(z) \). Mismatches between the estimates and actual values result in imperfect cancellation of noise from the first stage and deteriorates SNR. Thus, the number of stages that can practically be cascaded together is limited by the precision of the digital error correction.

**A. CASCADEN Tone \( \Sigma\Delta \) Converter**

If the stage-1 introduces zeros in the NTF at \( f_s/4N \), the noise-performance could be improved by placing zeros at the edge of the signal-band where the in-band noise is at its maximum. The easiest way of implementing this, if the signal bandwidth was \( f_s/8N \) would be to cascade an N-Tone and a 2N-Tone \( \Sigma\Delta \) Converter as shown in Figure 4. Noting that this nothing but a cascade of an N-Tone and 2N-Tone, the transfer function can be readily found to be \( Y(z) \).

\[ Y(z) = X(z) \cdot z^{-6N} + Eq_2 \cdot (1 + z^{-2N}) \cdot (1 + z^{-4N}) \]
The total quantization noise after noise shaping can be calculated as follows:

\[
|N(f)|^2 = 16\cos^4(2\pi N f / f_s)\cos^2(4\pi N f / f_s)
\]

\[
\begin{align*}
P_n &= \frac{2}{f_s} \int_{-(2n+1)f_s/4N-f_s/2}^{+(2n+1)f_s/4N+f_s/2} |\sigma_n^2| \, df \\
P_n &= \frac{2}{f_s} \int_{-(2n+1)f_s/4N-f_s/2}^{+(2n+1)f_s/4N+f_s/2} 16\cos^2(2\pi N f / f_s)\cos^2(4\pi N f / f_s)\sigma_n^2 \, df \\
P_n &\approx \frac{2}{f_s} \int_{-f_s/2}^{+f_s/2} 16(2\pi N f / f_s)^4\sigma_n^2 \, df
\end{align*}
\]

As can be seen for high OSR, this architecture provides performance similar to that of a first order N-Tone \(\Sigma\Delta\) converter. This is because at high OSRs, the zeros introduced by the 2N-Tone stage do not shape the in-band noise. At OSRs where the zeros introduced by the 2N-tone stage appears at the desired signal band-edge is when we see considerable improvement in SNR as compared to a first order \(\Sigma\Delta\) converter.

Shown in Figure 5 the overall transfer function is a product of the NTFs of an N-Tone \(\Sigma\Delta\) and a 2N-Tone \(\Sigma\Delta\). With an OSR of 16, simulations of the behavioral model in SIMULINK, shows the SNR achieved for a Cascaded N-2N Tone \(\Sigma\Delta\) Converter to be about 24dB.

**B. MASH N-Tone \(\Sigma\Delta\) Converter**

A special case of the cascaded \(\Sigma\Delta\) Converter, MASH implementations for N-tone \(\Sigma\Delta\) converter [7], [8], [9] cascade two first order N-Tone \(\Sigma\Delta\) Converters. This combines the higher-order noise-shaping with the robustness of a first-order modulator. Figure 6 shows the second order MASH-counterpart for the N-tone \(\Sigma\Delta\) converter [10]. The transfer function is given by (15) and the SNR is similar to that of a second-order modulator. With an OSR of 16, simulations of the behavioral model in SIMULINK, shows the SNR achieved for a second-order MASH to be about 34dB (Figure 7).

\[
Y(z) = X(z) \cdot z^{-2N} + E_q \cdot z^{-2N}(1 + z^{-2N})^2
\]

\[
|N(f)|^2 = 16\cos^4(2\pi N f / f_s) = 2^{(2n+1)f_s/4N+f_s/2}
\]

\[
P_n = \frac{2}{f_s} \int_{-(2n+1)f_s/4N-f_s/2}^{+(2n+1)f_s/4N+f_s/2} |(N(f))^2\sigma_n^2| \, df
\]

\[
P_n = \frac{2}{f_s} \int_{-(2n+1)f_s/4N-f_s/2}^{+(2n+1)f_s/4N+f_s/2} 16\cos^2(2\pi N f / f_s)\cos^2(4\pi N f / f_s)\sigma_n^2 \, df
\]

\[
P_n \approx \frac{2}{f_s} \int_{-f_s/2}^{+f_s/2} 16(2\pi N f / f_s)^4\sigma_n^2 \, df
\]

\[
SNR_{max} = 5.1 + 50\log(OSR) - 40\log(N)
\]

**C. Optimal Placement of Zeros of the NTF**

It is worthwhile to look at the variation of SNR with OSR as in Figure 8. In other words - the variation of SNR for a fixed NTF as the signal bandwidth is varied. For high OSR, the second-order MASH architecture is better than the Cascaded N-2N Tone architecture by atleast 12dB. This is not surprising, simply because the former has a two zeros at \(f_s/4N\) while the latter has a single zero at \(f_s/4N\). However, as OSR reduces, the zero introduced by the 2N-Tone \(\Sigma\Delta\) converter starts attenuating noise in the signal-band and the two performances become comparable. In fact, at OSR of
N-tone $\Sigma\Delta$ converters. While they are not sensitive to non-linearities in the DAC, the mismatches between the digital error correction and the analog circuits limit the performance. We also explored that the well-known fact of optimally spreading the zeros within the desired signal bandwidth to obtain a good SNR could be adapted to N-Tone $\Sigma\Delta$ converters.

**References**


